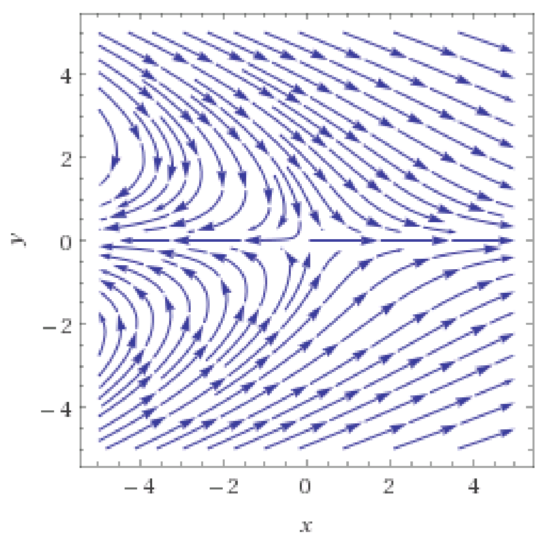
MATH 53H - Solutions to Problem Set V

1. For each of the three systems, the origin is the only equilibrium point.

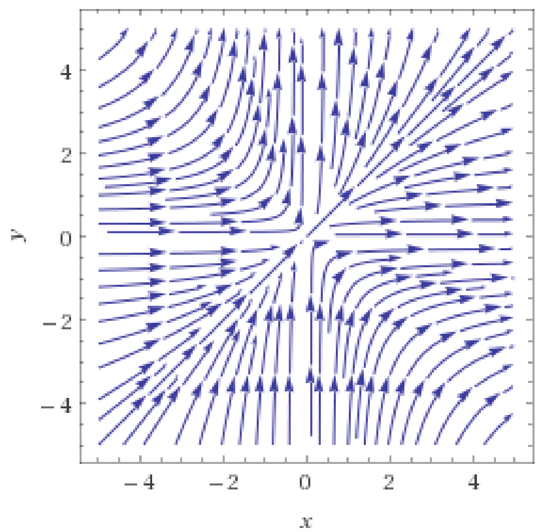
1. The associated linearized system is $x' = x,\ y' = -2y$ and hence the origin is a saddle. By solving for y first and then for x we obtain that

$$x(t) = (x_0 + \frac{{y_0}^2}{5})e^t - \frac{{y_0}^2}{5}e^{-4t},\ y(t) = y_0e^{-2t}$$

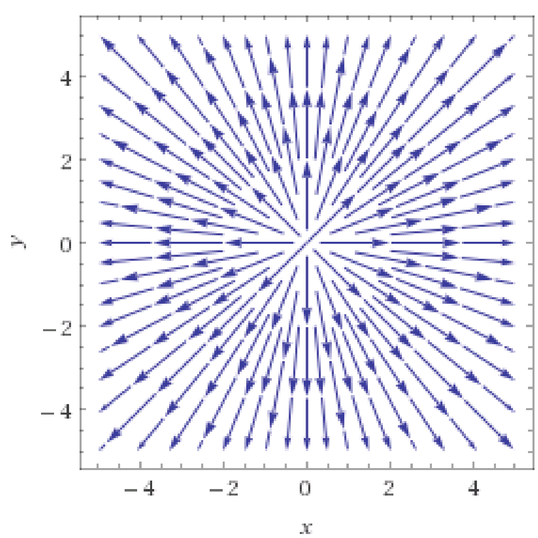
where we denote the initial condition by (x_0, y_0). We get in particular that $x(t) + \frac{y(t)^2}{5} = (x_0 + \frac{{y_0}^2}{5})e^t$ and by the parametrization of y, it is clear that the parabola $x = -\frac{y^2}{5}$ is the stable curve for the system.
By performing the change of variables $u = x + \frac{y^2}{5},\ v = y$ or just looking at the phase portrait, we can see that the linearized system describes accurately the behaviour near the origin.



2. The associated linearized system is $x' = 0,\ y' = 0$, which admits only constant solutions, and hence the origin is not a hyperbolic equilibrium point and there is no stable curve.
By looking at the phase portrait, we clearly see that the linearized system does not describe accurately the behaviour near the origin.



3. The associated linearized system is $x' = 0,\ y' = 0$, which admits only constant solutions, and hence the origin is not a hyperbolic equilibrium point and there is no stable curve.
By looking at the phase portrait, we clearly see that the linearized system does not describe accurately the behaviour near the origin. In order to draw the phase portrait or solve the system explicitly, it might be helpful to notice that in polar coordinates the system can be written as $r' = r^3,\ \theta' = 0$.



2. Our vector field is $F(x,y) = (y - x^5, -x - y^3)$. Hence we have

$$\langle \nabla L, F \rangle = 2x(y - x^5) + 2y(-x - y^3) = -2x^6 - 2y^4 \leq 0$$

with equality only at the origin. It is immediate from Lyapunov's theorems that the origin is asymptotically stable.

3. (i) This is algebraic manipulation using the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

(ii) In polar coordinates we need to show equivalently that $r(t) \to 1$ and $\theta(t) \to 2n\pi$ as $t \to \infty$ for some integer n.
We can solve the first equation for $r(t)$ by using separation of variables to obtain $\frac{r(t)}{|r(t)-1|} = \frac{r(0)}{|r(0)-1|}e^t$, which implies that $r(t) \to 1$ as $t \to \infty$ assuming $r(0) \neq 0$.
If $R(t)$ is an anti-derivative for $r(t)$, then we can use separation of variables again together with the double-angle formula $1 - \cos\theta = 2\sin^2(\frac{\theta}{2})$ to obtain $\theta(t) = 2\cot(R(t) + C)$ for some constant C. It is clear that since $r(t) \to \infty$, $R(t) \to \infty$ as $t \to \infty$, which implies that $\theta(t) \to 2n\pi$ as desired.

(iii) Note that if the initial condition is (r_0, θ_0), where for example we can take $r_0 < 1$, $\theta_0 > 0$ both close to 1 and 0 respectively, then since both $r(t)$, $\theta(t)$ are increasing by the equations of our system, we will have

$$\theta'(t) = r(t)(1 - \cos\theta(t)) \geq r_0(1 - \cos\theta_0) > 0$$

as long as $\theta(t)$ is say less than $\frac{\pi}{2}$. This clearly implies that for some time t_0 we will have $\theta(t_0) = \frac{\pi}{2}$ and hence the solution will move "far" from $(1, 0)$. Therefore $(1, 0)$ is not a stable equilibrium point.

4. You can find an argument along the lines of the hint in the textbook of the course (Brendle, Theorem 3.6, Lemmas 3.7-3.9, pages 32-35).